Subject: Mathematics (Abstract Algebra)

Topic: Group Theory - Definition & Examples

1 Formal Definition of a Group

A **Group** is a non-empty set G equipped with a binary operation $*$ (often denoted as multiplication or addition) such that the algebraic structure $(G, *)$ satisfies the following four axioms:

Definition 1 (Group Axioms). *For $(G, *)$ to be a group, the following conditions must hold:*

(G1) Closure Property: *For all elements $a, b \in G$, the result of the operation is also in G.*

$$\forall a, b \in G \implies a * b \in G$$

(G2) Associativity: *The order of operation does not matter.*

$$\forall a, b, c \in G \implies (a * b) * c = a * (b * c)$$

(G3) Existence of Identity: *There exists a unique element $e \in G$ (called the identity element) such that for any $a \in G$:*

$$a * e = a \quad \text{and} \quad e * a = a$$

(G4) Existence of Inverse: *For every $a \in G$, there exists an element $a^{-1} \in G$ (called the inverse) such that:*

$$a * a^{-1} = e \quad \text{and} \quad a^{-1} * a = e$$

2 Abelian Group (Commutative Group)

A group $(G, *)$ is called an **Abelian Group** if it satisfies an additional fifth axiom:

- **Commutativity:**

$$\forall a, b \in G \implies a * b = b * a$$

Note. If a group does not satisfy commutativity, it is called a **Non-Abelian Group**.

Set	**Operation**	**Identity (e)**	**Inverse (a^{-1})**	**Type**
Integers ($\mathbb{Z}$)	Addition ($+$)	0	$-a$	Infinite Abelian
Non-zero Reals ($\mathbb{R}^*$)	Multiplication ($\cdot$)	1	$1/a$	Infinite Abelian
Matrices $M_2(\mathbb{R})$	Addition ($+$)	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$-A$	Infinite Abelian
General Linear $GL(2, \mathbb{R})$	Matrix Mult. ($\cdot$)	I	A^{-1}	**Non-Abelian**

Table 1: Common Examples of Groups

3 Standard Examples

3.1 Detailed Analysis of Examples

Example (**The Group of Integers** $(\mathbb{Z}, +)$).

- **Closure:** Sum of two integers is an integer.
- **Identity:** $0 \in \mathbb{Z}$ such that $a + 0 = a$.
- **Inverse:** For $5 \in \mathbb{Z}$, inverse is -5 because $5 + (-5) = 0$.
- **Commutativity:** $a + b = b + a$ holds.
- **Conclusion:** Abelian Group.

Example (**General Linear Group** $GL(2, \mathbb{R})$). The set of 2×2 matrices with non-zero determinant.

- Matrix multiplication is associative but **not commutative** (generally $AB \neq BA$).
- **Conclusion:** Non-Abelian Group.

4 Important Non-Examples

1. **Natural Numbers** $(\mathbb{N}, +)$: Not a group because it lacks inverses (e.g., $-5 \notin \mathbb{N}$).
2. **Integers under Subtraction** $(\mathbb{Z}, -)$: Not a group because subtraction is not associative: $(5 - 3) - 2 \neq 5 - (3 - 2)$.
3. **Real Numbers under Multiplication** $(\mathbb{R}, \cdot)$: Not a group because 0 has no multiplicative inverse.

5 Elementary Properties (Theorems)

For any group $(G, *)$:

1. **Uniqueness of Identity:** The identity element is unique.
2. **Uniqueness of Inverse:** Every element has a unique inverse.

3. **Cancellation Laws:**

$$a * b = a * c \implies b = c \quad \text{(Left Cancellation)}$$

4. **Reversal Law (Socks-Shoes Property):**

$$(a * b)^{-1} = b^{-1} * a^{-1}$$