Metric Space: Definition and Examples

A **metric space** is an ordered pair (X, d), where X is a non-empty set and d : X × X → ■ is a function called a **metric** such that for all x, y, z ∈ X, the following axioms are satisfied:

- **Non-negativity:** d(x, y) ≥ 0
- **Identity of indiscernibles:** d(x, y) = 0 if and only if x = y
- **Symmetry:** d(x, y) = d(y, x)
- **Triangle inequality:** d(x, z) ≤ d(x, y) + d(y, z)

The concept of a metric space generalizes the idea of distance in geometry. It provides a mathematical framework to discuss notions such as convergence, continuity, and compactness in a very general setting.

Examples of Metric Spaces

Example 1: Real Numbers
Let X = ■ and define d(x, y) = |x – y|. Then (■, d) is a metric space. This is the standard metric used in real analysis.

Example 2: Euclidean Space ■■
Let X = ■■ and define d(x, y) = √[(x■ – y■)^2 + (x■ – y■)^2 + ... + (x■ – y■)^2]. This metric represents the usual distance between two points in n-dimensional space.

Example 3: Discrete Metric
Let X be any non-empty set and define d(x, y) = 0 if x = y, and d(x, y) = 1 if x ≠ y. This defines a metric called the discrete metric.

Example 4: Metric on Continuous Functions
Let X be the set of all continuous functions on [a, b]. Define d(f, g) = max |f(x) – g(x)| for x ∈ [a, b]. Then (X, d) forms a metric space.

Conclusion

Metric spaces form the foundation of modern analysis and topology. They allow mathematicians to extend geometric intuition to abstract spaces and are widely used in pure and applied mathematics.