Complex Analysis: Fundamentals with Detailed Examples

This document provides a detailed introduction to complex analysis, covering the algebra of complex numbers, the complex plane, complex polynomials, and power series. Each topic is explained step-by-step with illustrative examples, making it suitable for undergraduate mathematics and competitive exams.

1. Algebra of Complex Numbers

A complex number is defined as $z = a + ib$, where a and b are real numbers and i is the imaginary unit satisfying $i^2 = -1$. The real part of z is $\mathrm{Re}(z) = a$, and the imaginary part is $\mathrm{Im}(z) = b$.

Basic Operations

1 Addition: $(a + ib) + (c + id) = (a + c) + i(b + d)$

2 Multiplication: $(a + ib)(c + id) = (ac - bd) + i(ad + bc)$

3 Conjugate: z■ = a – ib

4 Modulus: $|z| = \sqrt{(a^2 + b^2)}$

Example 1: Let z■ = 2 + 3i and z■ = 1 – i. Then z■z■ = (2 + 3i)(1 – i) = 5 + i.

2. The Complex Plane

The complex plane (Argand plane) represents complex numbers geometrically. The horizontal axis represents the real part, while the vertical axis represents the imaginary part.

Polar Form

Any complex number $z \neq 0$ can be written in polar form as $z = r(\cos \theta + i \sin \theta)$, where $r = |z|$ and $\theta = \arg(z)$.

Example 2: For $z = 1 + i$, we have $r = \sqrt{2}$ and $\theta = \pi/4$. Thus $z = \sqrt{2} (\cos \pi/4 + i \sin \pi/4)$.

3. Polynomials of a Complex Variable

A complex polynomial is an expression of the form $P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0$, where the coefficients a_k are complex numbers.

The Fundamental Theorem of Algebra states that every non-constant complex polynomial has at least one complex root.

Example 3: Solve $z^2 + 1 = 0$. The roots are $z = \pm i$.

4. Power Series in Complex Analysis

A power series centered at z■ is an infinite series of the form ∑ a■ (z – z■)■. Power series play a central role in complex analysis because analytic functions can be represented locally by such series.

Radius of Convergence

There exists a radius R such that the power series converges for |z – z■| < R and diverges for |z – z■| > R.

Example 4: The geometric series ∑ z■ converges for $|z| < 1$ and diverges for $|z| \geq 1$.

Thus, algebraic structure, geometry, and infinite series together form the foundation of complex analysis.