Differential Geometry of Curves with Parameters Other Than Arc Length

In differential geometry, curves are often parameterized by arc length because it simplifies many formulas. However, in practical applications, curves are frequently described using parameters other than arc length, such as time or an arbitrary variable. This document explains how differential geometry concepts extend to such curves.

1. General Parametric Curve

Let a curve in three-dimensional space be defined as $r(t) = (x(t), y(t), z(t))$, where t is a general parameter and not necessarily the arc length. The velocity vector is $r'(t)$, and its magnitude $|r'(t)|$ gives the speed of the curve.

2. Arc Length and Reparameterization

The arc length s from $t = a$ to $t = b$ is given by $s = \int |r'(t)|\, dt$. If $|r'(t)| \neq 1$, then t is not the arc length parameter. The unit tangent vector is defined as $T = r'(t)/|r'(t)|$.

Example 1: Consider $r(t) = (t, t^2, 0)$. Then $r'(t) = (1, 2t, 0)$ and $|r'(t)| = \sqrt{(1 + 4t^2)}$. This shows that t is not the arc length parameter.

3. Curvature for Non-Arc Length Parameters

For a general parameter t, the curvature κ is given by $\kappa = |r'(t) \times r''(t)| / |r'(t)|^3$. This formula works regardless of the chosen parameter.

Example 2: For $r(t) = (a \cos t, a \sin t, 0)$, we have $r'(t) = (-a \sin t, a \cos t, 0)$ and $r''(t) = (-a \cos t, -a \sin t, 0)$. Substituting into the curvature formula gives $\kappa = 1/a$, which is constant as expected for a circle.

4. Torsion with General Parameters

The torsion τ of a space curve with a general parameter t is given by $\tau = \det(r'(t), r''(t), r'''(t)) / |r'(t) \times r''(t)|^2$. This measures how the curve twists out of its osculating plane.

Example 3: For a [redacted] (helix) $r(t) = (a \cos t, a \sin t, bt)$, torsion is constant and equal to $b/(a^2 + b^2)$. This demonstrates that torsion can be computed without arc length parameterization.

Thus, differential geometry concepts such as tangent, curvature, and torsion remain valid for curves parameterized by variables other than arc length, with appropriate formulas.