Dr. Amarendra Kumar
Deptt. of chemistry,
Jain college, Ara

Theorems of operators

Theorem ①

Two eigenfunctions of a Hermitian operator with different eigenvalues are orthogonal.

Proof ⇒

Let $\hat{A}$ be the Hermitian operator. Then its eigenvalue equations are

$$\hat{A}\psi_1 = \lambda_1\psi_1 \quad \text{---} \quad (1)$$

$$\hat{A}\psi_2 = \lambda_2\psi_2 \quad \text{---} \quad (2)$$

Where ψ_1 and ψ_2 are the eigenfunctions of $\hat{A}$ with λ_1 and λ_2 are the corresponding eigenvalues.

Taking the complex conjugate of eqn. ① and recalling that λ_1 is real.

$$\hat{A}^*\psi_1^* = \lambda_1\psi_1^* \quad \text{---} \quad (3)$$

Multiplying eqn ③ on the left by ψ_2 and integrating,

$$\int\psi_2\hat{A}^*\psi_1^* d\tau = \lambda_1\int\psi_2\psi_1^* d\tau \quad \text{---} \quad (4)$$

Again, Multiplying equation ② on the left by ψ_1^* and integrating,

$$\int\psi_1^*\hat{A}\psi_2 d\tau = \lambda_2\int\psi_1^*\psi_2 d\tau \quad \text{---} \quad (5)$$

Since $\hat{A}$ is a Hermitian operator, the left hand sides of equation ④ and ⑤ are equal. So that

$$\lambda_1\int\psi_2\psi_1^* d\tau = \lambda_2\int\psi_1^*\psi_2 d\tau$$

or, $$(\lambda_1 - \lambda_2)\int\psi_1^*\psi_2 d\tau = 0 \quad \text{---} \quad (6)$$

Since,

$$\lambda_1 \neq \lambda_2, \quad \int\psi_1^*\psi_2 d\tau = 0$$

Thus,

ψ_1 and ψ_2 are orthogonal.

Theorem (2)

If two operators commute, they have the same set of eigenfunctions.

Proof $\Rightarrow$

Let $\hat{A}$ be the operator whose eigenvalue equation is

$$\hat{A}\psi_i = a_i\psi_i \quad \text{——— (1)}$$

Where ψ_is are the set of eigenfunctions with a_is as the corresponding eigenvalues.

Since $\hat{A}$ and $\hat{B}$ commute, hence,

$$\hat{A}\hat{B} = \hat{B}\hat{A} \quad \text{——— (2)}$$

operating on ψ_i,

$$\hat{A}\hat{B}\psi_i = \hat{B}\hat{A}\psi_i = \hat{B}(a_i\psi_i) = a_i(\hat{B}\psi_i) \quad \text{——— (3)}$$

This shows that $(\hat{B}\psi_i)$ is an eigenfunction of $\hat{A}$ with eigenvalue a_i. This is possible only if $(\hat{B}\psi_i)$ is a multiple of ψ_i

i.e.

$$\hat{B}\psi_i = b_i\psi_i \quad \text{——— (4)}$$

In other words ψ_i is also an eigenfunction of $\hat{B}$.

—x—

SM