

 Find out the expression for the classical energy of a simple harmonic oscillator.

**Ans.** According to Hooke's law, the restoring force ( $F$ ) acting on a simple harmonic oscillator of mass  $m$  is directly proportional to its displacement  $x$  from the mean position, i.e.,

$$F = -kx \quad \dots (1)$$

Here  $k$  is a positive constant, called force-constant. The negative sign indicates that the direction of the force is always opposite to the direction of displacement.

According to the Newton's second law of motion, we have

$$F = m \cdot \frac{d^2x}{dt^2} \quad \dots (2)$$

From equations (1) and (2), we have

$$m \cdot \frac{d^2x}{dt^2} = -kx$$

$$\text{or, } \frac{d^2x}{dt^2} = -\left(\frac{k}{m}\right) \cdot x$$

$$\text{or, } \frac{d^2x}{dt^2} = -\omega^2x, \text{ where } \omega = \sqrt{\frac{k}{m}}$$

$$\text{or, } \frac{d^2x}{dt^2} + \omega^2x = 0 \quad \dots(3)$$

The general solution of equation (3) is

$$x = A \sin(\omega t + \theta) \quad \dots(4)$$

Where  $A$  and  $\theta$  are constants which can be determined from initial conditions. Equation (4) shows that the particle performs simple harmonic oscillations with frequency

$$v = \frac{\omega}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{k}{m}} \quad \dots(5)$$

where  $\omega$  is termed as the angular frequency of oscillations.

The potential energy,  $V(x)$ , of the oscillator at distance  $x$  from the mean position is given by

$$V(x) = - \int_0^x F \cdot dx$$

$$\text{or, } V(x) = - \int_0^x -kx \cdot dx \quad (\because F = -kx)$$

$$\text{or, } V(x) = k \int_0^x x dx = \frac{1}{2} kx^2 = \frac{1}{2} m\omega^2 x^2 \quad \left( \because \omega = \sqrt{\frac{k}{m}} \right) \quad \dots(6)$$

The kinetic energy of the oscillator at distance  $x$  from the mean position is given by

$$k(x) = \frac{1}{2} m \cdot \left( \frac{dx}{dt} \right)^2 = \frac{1}{2} mA^2\omega^2 \cos^2(\omega t + \theta)$$

$$= \frac{1}{2} mA^2\omega^2 [1 - \sin^2(\omega t + \theta)]$$

$$= \frac{1}{2} m\omega^2 (A^2 - x^2) \quad \dots(7)$$

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Therefore, according to the classical mechanics, the total energy ( $E$ ) of the oscillator is given by

$$\begin{aligned} E &= k(x) + V(x) \\ &= \frac{1}{2} m\omega^2 (A^2 - x^2) + \frac{1}{2} m\omega^2 x^2 \\ &= \frac{1}{2} m\omega^2 A^2 \end{aligned} \quad \dots(5)$$

As per the classical mechanics, the particle may oscillate with any amplitude ( $A$ ), consequently the total energy ( $E$ ) of oscillator increases continuously with the increase in amplitude. Further, the total energy of oscillator, oscillating with a constant frequency,  $\nu = \frac{\omega}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{k}{m}}$ , is proportional to the square of the amplitude. It also implies from equation (6) that the particle oscillates in a parabolic potential well.