

Hermite Differential Equation : The Schrodinger equation for one dimensional simple harmonic oscillator is given by equation (4) as

$$\frac{d^2\psi}{dx^2} + \left(\frac{2mE}{\hbar^2} - \frac{m^2\omega^2}{\hbar^2} \cdot x^2 \right) \psi = 0$$

Let us introduce a dimensionless independent variable y which is related to x by the equation,

$$y = \sqrt{\frac{m\omega}{\hbar}} \cdot x \quad \dots(5)$$

or,

$$x = \sqrt{\frac{\hbar}{m\omega}} \cdot y$$

Now, we have

$$\frac{d\psi}{dx} = \frac{d\psi}{dy} \cdot \frac{dy}{dx} = \frac{d\psi}{dy} \cdot \sqrt{\frac{m\omega}{\hbar}}$$

and
$$\frac{d^2\Psi}{dx^2} = \frac{d^2\Psi}{dy^2} \cdot \frac{dy}{dx} \sqrt{\frac{m\omega}{\hbar}} = \frac{d^2\Psi}{dy^2} \cdot \frac{m\omega}{\hbar} = \frac{m\omega}{\hbar} \cdot \frac{d^2\Psi}{dy^2}$$

Substituting the value of $\frac{d^2\Psi}{dx^2}$ and x^2 in equation (4), we get

$$\frac{m\omega}{\hbar} \cdot \frac{d^2\Psi}{dy^2} + \left(\frac{2mE}{\hbar^2} - \frac{m^2\omega^2}{\hbar^2} \cdot \frac{\hbar}{m\omega} \cdot y^2 \right) \Psi = 0$$

or,
$$\frac{m\omega}{\hbar} \cdot \frac{d^2\Psi}{dy^2} + \left(\frac{2mE}{\hbar^2} - \frac{m\omega}{\hbar} \cdot y^2 \right) \Psi = 0$$

or,
$$\frac{m\omega}{\hbar} \cdot \frac{d^2\Psi}{dy^2} + \frac{m\omega}{\hbar} \left(\frac{\hbar}{m\omega} \cdot \frac{2mE}{\hbar^2} - y^2 \right) \Psi = 0$$

or,
$$\frac{d^2\Psi}{dy^2} + \left(\frac{2E}{\hbar\omega} - y^2 \right) \Psi = 0 \quad \dots (7)$$

or,
$$\frac{d^2\Psi}{dy^2} + (\lambda - y^2) \Psi = 0, \text{ where } \lambda = \frac{2E}{\hbar\omega} \quad \dots (8)$$

For large values of y , such that $y^2 \gg \lambda$, we may neglect λ . Thus the equation (8) is changed to

$$\frac{d^2\Psi}{dy^2} - y^2\Psi = 0 \quad \dots (9)$$

For large values of y , the approximate solution of equation (9)

$$\Psi = e^{-y^2/2} \quad \dots (10)$$

Since $\Psi \rightarrow 0$, when $y \rightarrow \infty$, the solution of the form $\Psi = e^{+y^2/2}$ is rejected.

If we substitute, $\Psi = e^{-y^2/2}$, in equation (9), we get

$$\frac{d^2\Psi}{dy^2} - (y^2 - 1) \Psi = 0 \quad \dots (11)$$

For large values of y , this equation is reduced to equation (9). This suggests that an accurate solution of equation (7) must be of the form

$$\Psi = e^{-y^2/2} \cdot H(y) \quad \dots (12)$$

Where $H(y)$ is a finite polynomial in y .

Differentiating equation (12), with respect to y , we get

$$\frac{d\psi}{dy} = e^{-y^2/2} \cdot \frac{dH(y)}{dy} + H(y) \cdot e^{-y^2/2} \cdot (-y)$$

$$= \left[\frac{dH(y)}{dy} - y \cdot H(y) \right] e^{-y^2/2}$$

$$\begin{aligned} \text{or, } \frac{d^2\psi}{dy^2} &= \left[\frac{d^2H(y)}{dy^2} - y \frac{dH(y)}{dy} - H(y) \right] e^{-y^2/2} \\ &\quad + \left[\frac{dH(y)}{dy} - H(y) \cdot y \right] \cdot e^{-y^2/2} \cdot (-y) \\ &= \left[\frac{d^2H(y)}{dy^2} - 2y \cdot \frac{dH(y)}{dy} - H(y) + H(y) \cdot y^2 \right] e^{-y^2/2} \end{aligned}$$

Now substituting the expression for $\frac{d^2\psi}{dy^2}$ and the expression for ψ in equation (7), we get

$$\left[\frac{d^2H(y)}{dy^2} - 2y \cdot \frac{dH(y)}{dy} - H(y) + H(y) \cdot y^2 \right] e^{-y^2/2} + (\lambda - y^2) \cdot H(y) \cdot e^{-y^2/2} = 0$$

$$\text{or, } \frac{d^2H(y)}{dy^2} - 2y \cdot \frac{dH(y)}{dy} + (\lambda - 1) H(y) = 0 \quad \dots(13)$$

This is well-known Hermite's differential equation.