Uniqueness of Series Representation and Convergence of Power Series

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1. Power Series

A power series centered at a point a is an infinite series of the form $\Sigma a_n (z - a)^n$, where a_n are constants and z is a complex variable.

2. Uniqueness of Series Representation

Theorem: If a power series $\Sigma a_n (z - a)^n$ converges to zero for all z in some neighborhood of a, then all coefficients $a_n = 0$.

Proof: Suppose $\Sigma a_n (z - a)^n = 0$ for all z in some open disk centered at a. Since power series are analytic within their radius of convergence, we may differentiate term by term. Evaluating at $z = a$ gives $a_0 = 0$. Differentiating once and evaluating at $z = a$ gives $a_1 = 0$. Proceeding inductively, we obtain $a_n = 0$ for all n. Hence the representation is unique.

Corollary: If two power series are equal in a neighborhood of a point, then their corresponding coefficients are equal.

3. Absolute Convergence of Power Series

Theorem: A power series $\Sigma a_n (z - a)^n$ converges absolutely inside its radius of convergence R.

Proof: By Cauchy-Hadamard theorem, the radius of convergence is given by $1/R = \limsup |a_n|^{1/n}$. For $|z - a| < R$, we can choose r such that $|z - a| < r < R$. Then $|a_n (z - a)^n| \leq |a_n| r^n$. Since $\Sigma |a_n| r^n$ converges, comparison test implies $\Sigma |a_n (z - a)^n|$ converges. Hence the series converges absolutely.

Example:

Consider $\Sigma z^n / n!$. Using ratio test, $R = \infty$. Thus the series converges absolutely for all $z \in C$. This is the exponential series.

4. Uniform Convergence of Power Series

Theorem: A power series converges uniformly on every closed disk $|z - a| \leq r$, where $r < R$ (R is radius of convergence).

Proof: If $|z - a| \leq r < R$, then $|a_n (z - a)^n| \leq |a_n| r^n$. Since $\Sigma |a_n| r^n$ converges, by Weierstrass M-test the series converges uniformly on the closed disk.

Example:

Consider Σz^n. The radius of convergence is $R = 1$. The series converges uniformly on $|z| \leq r$ for any $r < 1$, but not uniformly on $|z| < 1$ as a whole.

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